

## Exercise Sheet 11

**Exercise 1.** Let  $X$  be a path-connected topological space and fix  $x_0 \in X$ . A loop  $\gamma : S^1 \rightarrow X$  represents an element of  $\pi_1(X, x_0)$  and determines a class in  $H_1(X, \mathbb{Z})$ . Define

$$\Phi_{x_0} : \pi_1(X, x_0) \rightarrow H_1(X, \mathbb{Z}), \quad \Phi_{x_0}([\gamma]) = [\gamma] \in H_1(X, \mathbb{Z}),$$

where on the right-hand side we write  $\gamma$  as shorthand for the induced 1-cycle  $\gamma_{\#}(\sigma)$  for a fixed 1-cycle  $\sigma$  of  $S^1$ .

- (1) Show that  $\Phi_{x_0}$  is a well-defined group homomorphism.
- (2) If  $x_1 \in X$  and  $\eta$  is a path from  $x_0$  to  $x_1$ , let  $c_{\eta}([\gamma]) = [\eta^{-1} * \gamma * \eta] \in \pi_1(X, x_1)$ . Show that the maps to homology are compatible with basepoint change, that is,  $\Phi_{x_1} \circ c_{\eta} = \Phi_{x_0}$ .
- (3) Deduce that the map  $\Phi_{x_0}$  induces an isomorphism

$$\pi_1(X, x_0)^{\text{ab}} \cong H_1(X, \mathbb{Z}),$$

where  $\pi_1(X)^{\text{ab}} = \pi_1(X)/[\pi_1(X), \pi_1(X)]$  is the abelianization.

**Exercise 2.** (for credit, due on 7 December)

Consider the unit square  $[0, 1]^2$  with sides identified according to  $aba^{-1}b^{-1}$ . Let  $\pi : [0, 1]^2 \rightarrow T^2$  be the quotient map. Identify  $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}\langle [a] \rangle \oplus \mathbb{Z}\langle [b] \rangle$ , where  $[a]$  is represented by the image of the bottom edge oriented left-to-right, and  $[b]$  by the right edge oriented bottom-to-top. For each path  $\alpha$  below, compute the homology class of the loop  $\pi \circ \alpha$ .

- (1) (1 point) The diagonal from  $(0, 0)$  to  $(1, 1)$ .
- (2) (1 point) The diagonal from  $(1, 0)$  to  $(0, 1)$ .
- (3) (1 point) The loop obtained by traversing the three straight segments

$$(0, 0) \rightarrow (1, 0) \rightarrow (1, 1) \rightarrow (0, 0).$$

- (4) (2 points) Fix coprime integers  $p, q \in \mathbb{Z}_{\geq 1}$  and define

$$\alpha_{p,q}(t) = (pt \bmod 1, qt \bmod 1), \quad t \in [0, 1].$$

Show that  $\pi \circ \alpha_{p,q}$  is a loop with no self-intersections and compute its class in  $H_1(T^2, \mathbb{Z})$ .

**Exercise 3.** Show that if  $\alpha \sim \alpha + \partial C$  in  $H_1$ , then  $(\alpha + \partial C) \circ \beta = \alpha \circ \beta$  for every 1-cycle  $\beta$ .

**Exercise 4.** Let  $X$  be a Riemann surface of genus  $g$ . Recall that  $X$  has a planar model with symbol  $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$ . We use the same symbols  $a_j, b_j$  to denote the corresponding homology classes in  $H_1(X, \mathbb{Z})$ . Denote by  $\circ$  the intersection pairing on  $H_1(X, \mathbb{Z})$ .

- (1) Show using the presentation

$$\pi_1(X) \cong \left\langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \right\rangle$$

that

$$H_1(X, \mathbb{Z}) \cong \pi_1(X)^{\text{ab}} \cong \mathbb{Z}^{2g}.$$

- (2) Show that

$$a_i \circ b_j = \delta_{ij}, \quad a_i \circ a_j = 0 = b_i \circ b_j \text{ for all } i, j.$$

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- (3) A canonical basis of  $H_1(X, \mathbb{Z})$  is a basis  $(a_1, \dots, a_g, b_1, \dots, b_g)$  satisfying the identities in (2). Let  $a'_1, \dots, a'_g, b'_1, \dots, b'_g$  be another canonical basis of  $H_1(X, \mathbb{Z})$ . Show that there is a matrix  $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$  such that

$$\begin{pmatrix} a'_1 \\ \vdots \\ a'_g \\ b'_1 \\ \vdots \\ b'_g \end{pmatrix} = M \begin{pmatrix} a_1 \\ \vdots \\ a_g \\ b_1 \\ \vdots \\ b_g \end{pmatrix}.$$